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Generalized orthopair fuzzy matrices based on Hamacher operations

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Abstract. The objective of this paper is to apply the concept of intuitionistic fuzzy matrices and pythagorean fuzzy matrices to q-rung orthopair fuzzy matrices. In this paper, we introduce the Hamacher operations of q-rung orthopair fuzzy matrices and prove some desirable properties of these operations, such as commutativity, idempotency and monotonicity. Further, we prove De Morgan's laws over complement for these operations. Then we constructed the scalar multiplication $(n._h A)$ and exponentiation $(A^{\wedge_h n})$ operations of q-rung orthopair fuzzy matrices and investigate their algebraic properties. Finally, we prove some properties of necessity and possibility operators of q-rung orthopair fuzzy matrices.

1. Introduction

The concept of an intuitionistic fuzzy matrix (IFM) was introduced by Khan et al. [3] and simultaneously Im et al. [2] to generalize the concept of Thomason's [28] fuzzy matrix. Each element in an IFM is expressed by an ordered pair $\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle$. The sum $\mu_{a_{ij}} + \nu_{a_{ij}}$ of each ordered pair is less than or equal to 1. Further, Emam and Fndh [1] defined some kinds of IFMs, also they construct an idempotent IFM from any given one through the min-max composition. Pal [9] introduced the intuitionistic fuzzy determinant and [4] defined some basic operations and relations of IFMs including maxmin, minmax, complement, algebraic sum, algebraic product etc. and proved equality between IFMs. Mondal and Pal [6] studied the similarity relations, together with invertibility conditions and eigenvalues of IFMs. Zhang and Xu [31] studied intuitionistic fuzzy value and introduced the concept of composition two intuitionistic fuzzy matrices. Muthuraji et al. [7] obtained a decomposition of intuitionistic fuzzy matrices.

Yager [29] introduced the concept of the Pythagorean fuzzy set (PFS) and developed some aggregation operations for PFS. Pend and Yang [10] proved

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some results on pythagorean fuzzy sets. Zhang and Xu [32] studied various binary operations over PFS and also proposed a decision making algorithm based on PFS. In [16] Using the theory of PFS, we defined the Pythagorean fuzzy matrix (PFM) and its algebraic operations, they constructed nA and A^n of a Pythagorean fuzzy matrix A and investigated their algebraic properties. Then [20] we have developed Hamacher operations on Pythagorean fuzzy matrices and investigates their algebraic properties. Further [21] we constructed $n._h A$ and $A^{\wedge_h n}$ of a Pythagorean fuzzy matrix A and investigated their algebraic properties. Recently, Yager [30] again proposed the concept of the q-ROFS, in which MD u and NMD satisfy $\mu^q + \nu^q \leq 1$ $(q \geq 1)$. We can see that the IFS and PFS are special cases of q-ROFS. As q-rung increases, the range of processing fuzzy information increases. In recent years, the topic of information aggregation has attracted a lot of attention and is one of the key research issues in the problems of MAGDM. As far as q-ROFS is concerned, different aggregation operators have been introduced and applied, such as q-ROFWA and q-ROFWG operator [5]. In other words, IFS and PFS are two special forms of q-ROFS, which means that the q-ROFSs are able to handle higher levels of uncertainties. Using the theory of q-ROFS, we introduced the q-rung orthopair fuzzy matrix (PFM) and its algebraic operations are defined [26]. Then we constructed nA and A^n of a q-rung orthopair fuzzy matrix A and investigated their algebraic properties. Since the appearance of Fuzzy matrix, IFM, PFM, Fermatean fuzzy matrix theory, several researchers have importantly contributed to the development of this theory and its applications (see, for example, [8, 11–15, 17–19, 22–25, 27]). The focus of this paper, we have developed the Hamacher operations of q-rung orthopair fuzzy matrices and proved their algebraic properties.

This paper is organized as follows: In Section 2, we shall develop the Hamacher operations of q-rung orthopair fuzzy matrices and analyze some desirable properties. In section 3, some results on complement of q-rung orthopair fuzzy matrices. In section 4, we constructed Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations of q-rung orthopair fuzzy matrix A and investigated their algebraic properties. In section 5, we prove some properties of necessity and possibility operators on q-rung orthopair fuzzy matrices. Section 6 concludes the paper with some future directions.

We introduce the Hamacher operations of q-rung orthopair fuzzy matrices and examples are given.

2. Hamacher operations of q-rung orthopair fuzzy matrices

In this section, we define the Hamacher operations of q-rung orthopair fuzzy matrices and then analyze some desirable properties these operations.

Definition 1. Let $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right]$ and $B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$ be any two q-rung orthopair fuzzy matrices of the same size, then we have:

(i) The Hamacher sum of A and B is defined by $A \boxplus_h B = (\alpha_{ij})$, where

$$\alpha_{ij} = \begin{cases} \langle 1, 0 \rangle, \text{ if } \left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle = \langle 1, 0 \rangle, \left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle = \langle 1, 0 \rangle, \\ \left\langle \left(\dfrac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{1 - \mu^q_{a_{ij}}\mu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\dfrac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}}\nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle, \quad \text{otherwise}, \end{cases}$$

for all i, j, and

(ii) The Hamacher product of A and B is defined by $A \boxdot_h B = (\beta_{ij})$, where

$$\beta_{ij} = \begin{cases} \langle 0, 1 \rangle, \text{ if } \left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle = \langle 0, 1 \rangle, \left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle = \langle 0, 1 \rangle, \\ \left\langle \left(\dfrac{\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}}\mu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\dfrac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}}\nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle, \quad \text{otherwise}, \end{cases}$$

for all i, j.

Example 1. For understanding the q-ROFM better, we give an instance to illuminate the understandability of the q-ROFM.

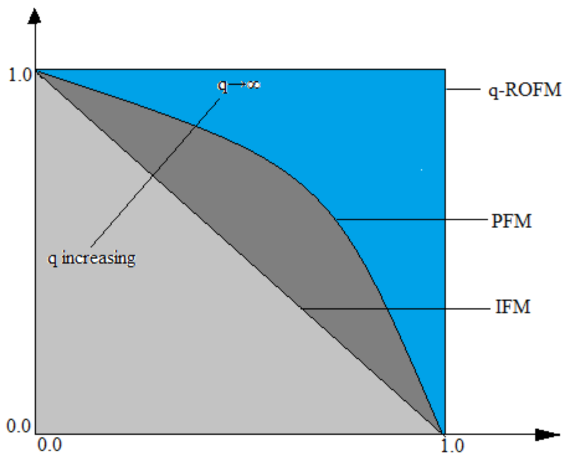

FIGURE 1. The Comparison of grades space of IFM, PFM and q-ROFM.

We can definitely get $0.9 + 0.6 > 1$, and, therefore, it does not follow the condition of intuitionistic fuzzy matrices. Also, we can get $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 > 1$, which does not obey the constraint condition of q-rung orthopair fuzzy set. However, we can get $(0.9)^q + (0.6)^q \leq 1 (q \geq 1)$. which is good enough to apply the q-ROFM to control it. This development can be evidently recognized in Figure 1. Here we notice that IFMs are all points beneath the line $\mu_{a_{ij}} + \nu_{a_{ij}} \leq 1$, the q-ROFMs are all points with $\mu^2_{a_{ij}} + \nu^2_{a_{ij}} \leq 1$, and the q-ROFMs are all points with $\mu^q_{a_{ij}} + \nu^q_{a_{ij}} \leq 1$ $(q \geq 1)$.

We see then that the q-ROFMs enable for the presentation of a bigger body of nonstandard membership function then IFMs and PFMs.

Lemma 1. *For any two real numbers $a, b \in [0,1]$, the following inequality holds*

$$\left(\frac{a^q b^q}{a^q + b^q - a^q b^q}\right)^{\frac{1}{q}} \leq \left(\frac{a^q + b^q - 2a^q b^q}{1 - a^q b^q}\right)^{\frac{1}{q}}.$$

Lemma 2. *For any three real numbers $a, b, c \in [0,1]$, the following inequalities hold. If $a \leq b$ then*

(i) $$\left(\frac{a^q c^q}{a^q + c^q - a^q c^q}\right)^{\frac{1}{q}} \leq \left(\frac{b^q c^q}{b^q + c^q - b^q c^q}\right)^{\frac{1}{q}},$$

(ii) $$\left(\frac{a^q + c^q - 2a^q c^q}{1 - a^q c^q}\right)^{\frac{1}{q}} \leq \left(\frac{b^q + c^q - 2b^q c^q}{1 - b^q c^q}\right)^{\frac{1}{q}}.$$

The relation between Hamacher sum and Hamacher product is established by the following theorem.

Theorem 1. *Let $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right], B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$ be any two q-ROFMs of the same size, then $A \boxdot_h B \leq A \boxplus_h B$.*

Proof. By using Lemma 1,

$$\left(\frac{\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}} \mu^q_{b_{ij}}}\right)^{\frac{1}{q}} \leq \left(\frac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{1 - \mu^q_{a_{ij}} \mu^q_{b_{ij}}}\right)^{\frac{1}{q}}$$

and

$$\left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}}\right)^{\frac{1}{q}} \geq \left(\frac{\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}} \nu^q_{b_{ij}}}\right)^{\frac{1}{q}},$$

for all i, j.

By Definition 1, it follows that $A \boxdot_h B \leq A \boxplus_h B$. □

Theorem 2. *For any q-rung orthopair fuzzy matrix A,*

(i) $A \boxplus_h A \geq A$,

(ii) $A \boxdot_h A \leq A$.

Proof. (i)

$$A \boxplus_h A = \left[\left\langle \left(\frac{2\mu^q_{a_{ij}} - 2a^q_{ij}}{1 - a^q_{ij}}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}{}^q}{2\nu^q_{a_{ij}} - \nu_{a_{ij}}{}^q}\right)^{\frac{1}{q}} \right\rangle\right]$$

$$= \left[\left\langle \left(\frac{2\mu^q_{a_{ij}}}{1 + \mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{2 - \nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right] \geq \left[\left\langle \mu^q_{a_{ij}}, \nu^q_{a_{ij}} \right\rangle\right].$$

Since

$$\mu^q_{a_{ij}} \leq \frac{2\mu^q_{a_{ij}}}{1 + \mu^q_{a_{ij}}} \quad \text{and} \quad \nu^q_{a_{ij}} \geq \frac{\nu^q_{a_{ij}}}{2 - \nu^q_{a_{ij}}},$$

$$A \boxplus_h A \geq \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right], \text{ for all } i, j$$
$$\geq A.$$

(ii) It can be proved similarly. □

The following theorem is obvious. The operations Hamacher sum and Hamacher product of q-ROFMs are commutative as well as associative, and the identities for $\boxplus_h$ and $\boxdot_h$ exist.

Theorem 3. *Let* $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right]$, $B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$, $C = \left[\left\langle \mu_{c_{ij}}, \nu_{c_{ij}} \right\rangle\right]$, *be any two q-ROFMs of the same size, then*

(i) $A \boxplus_h B = B \boxplus_h A$,
(ii) $(A \boxplus_h B) \boxplus_h C = A \boxplus_h (B \boxplus_h C)$,
(iii) $A \boxdot_h B = B \boxdot_h A$,
(iv) $(A \boxdot_h B) \boxdot_h C = A \boxdot_h (B \boxdot_h C)$.

Definition 2 ([31])**.** The $m \times n$ zero q-ROFM O is a q-ROFM all of whose entries are $\langle 0, 1\rangle$. The $m \times n$ universal q-ROFM J is a q-ROFM all of whose entries are $\langle 1, 0\rangle$.

Theorem 4. *For any q-rung orthopair fuzzy matrix A, then*

(i) $A \boxplus_h O = O \boxplus_h A = A$,
(ii) $A \boxdot_h J = J \boxdot_h A = A$,
(iii) $A \boxplus_h J = J$,
(iv) $A \boxdot_h O = O$.

The set of all q-ROFMs with respect to the Hamacher sum and Hamacher product form a commutative monoid. The Hamacher operations do not obey De Morgan's laws over transpose.

Theorem 5. *Let* $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right], B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$ *be any two q-ROFMs of the same size, then*

(i) $(A \boxplus_h B)^T = A^T \boxplus_h B^T$
(ii) $(A \boxdot_h B)^T = A^T \boxdot_h B^T$,

where A^T is the transpose of A.

Theorem 6. *Let* $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right], B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$ *be any two q-ROFMs of the same size, if $A \leq B$, then $A \boxdot_h C \leq B \boxdot_h C$.*

Proof. Let $\mu_{a_{ij}} \leq \mu_{b_{ij}}$ and $\nu_{a_{ij}} \geq \nu_{b_{ij}}$, for all i, j. By using Lemma 2,

$$\left(\frac{\mu^q_{a_{ij}} \mu^q_{c_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{c_{ij}} - \mu^q_{a_{ij}} \mu^q_{c_{ij}}}\right)^{\frac{1}{q}} \leq \left(\frac{\mu^q_{b_{ij}} \mu^q_{c_{ij}}}{\mu^q_{b_{ij}} + \mu^q_{c_{ij}} - \mu^q_{b_{ij}} \mu^q_{c_{ij}}}\right)^{\frac{1}{q}},$$

$$\left(\frac{\nu^q_{b_{ij}} \nu^q_{c_{ij}}}{\nu^q_{b_{ij}} + \nu^q_{c_{ij}} - \nu^q_{b_{ij}} \nu^q_{c_{ij}}}\right)^{\frac{1}{q}} \geq \left(\frac{\nu^q_{a_{ij}} \nu^q_{c_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{c_{ij}} - \nu^q_{a_{ij}} \nu^q_{c_{ij}}}\right)^{\frac{1}{q}},$$

for all i, j.

Therefore, $A \boxdot_h C \leq B \boxdot_h C$. □

Theorem 7. *Let $A = [\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle], B = [\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle]$ be any two q-ROFMs of the same size, if $A \leq B$, then $A \boxplus_h C \leq B \boxplus_h C$.*

Proof. Let $\mu_{a_{ij}} \leq \mu_{b_{ij}}$ and $\nu_{a_{ij}} \geq \nu_{b_{ij}}$, for all i, j. By using Lemma 2,

$$\left(\frac{\mu_{a_{ij}}^q + \mu_{c_{ij}}^q - 2\mu_{a_{ij}}^q \mu_{c_{ij}}^q}{1 - \mu_{a_{ij}}^q \mu_{c_{ij}}^q}\right)^{\frac{1}{q}} \leq \left(\frac{\mu_{b_{ij}}^q + \mu_{c_{ij}}^q - 2\mu_{b_{ij}}^q \mu_{c_{ij}}^q}{1 - \mu_{b_{ij}}^q \mu_{c_{ij}}^q}\right)^{\frac{1}{q}},$$

$$\left(\frac{\nu_{b_{ij}}^q + \nu_{c_{ij}}^q - 2\nu_{b_{ij}}^q \nu_{c_{ij}}^q}{1 - \nu_{b_{ij}}^q \nu_{c_{ij}}^q}\right)^{\frac{1}{q}} \geq \left(\frac{\nu_{a_{ij}}^q + \nu_{c_{ij}}^q - 2\nu_{a_{ij}}^q \nu_{c_{ij}}^q}{1 - \nu_{a_{ij}}^q \nu_{c_{ij}}^q}\right)^{\frac{1}{q}}.$$

Therefore, $A \boxplus_h C \leq B \boxplus_h C$. □

Definition 3. Let $A = [\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle]$ and $B = [\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle]$ be two q-ROFMs of the same size, then we have:

(i) $A \vee B = [\langle \max\{\mu_{a_{ij}}, \mu_{b_{ij}}\}, \min\{\nu_{a_{ij}}, \nu_{b_{ij}}\} \rangle]$,
(ii) $A \wedge B = [\langle \min\{\mu_{a_{ij}}, \mu_{b_{ij}}\}, \max\{\nu_{a_{ij}}, \nu_{b_{ij}}\} \rangle]$,
(iii) $A^C = [\langle \nu_{a_{ij}}, \mu_{a_{ij}} \rangle]$.

Theorem 8. *Let $A = [\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle], B = [\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle]$ be any two q-ROFMs of the same size, then*

(i) $(A \wedge B) \boxplus_h (A \vee B) = A \boxplus_h B$,
(ii) $(A \wedge B) \boxdot_h (A \vee B) = A \boxdot_h B$.

Proof. (i) $(A \wedge B) \boxplus_h (A \vee B)$

$$= \left[(\min\{\mu_{a_{ij}}, \mu_{b_{ij}}\}, \max\{\nu_{a_{ij}}, \nu_{b_{ij}}\}) \boxplus_h (\max\{\mu_{a_{ij}}, \mu_{b_{ij}}\}, \min\{\nu_{a_{ij}}, \nu_{b_{ij}}\})\right]$$

$$= \left(\frac{\min\left\{\mu_{a_{ij}}^q, \mu_{b_{ij}}^q\right\} + \max\left\{\mu_{a_{ij}}^q, \mu_{b_{ij}}^q\right\} - 2\min\left\{\mu_{a_{ij}}^q, \mu_{b_{ij}}^q\right\}\max\left\{\mu_{a_{ij}}^q, \mu_{b_{ij}}^q\right\}}{1 - \min\left\{\mu_{a_{ij}}^q, \mu_{b_{ij}}^q\right\}\max\left\{\mu_{a_{ij}}^q, \mu_{b_{ij}}^q\right\}}\right)^{\frac{1}{q}},$$

$$\left(\frac{\max\left\{\nu_{a_{ij}}^q, \nu_{b_{ij}}^q\right\}\min\left\{\nu_{a_{ij}}^q, \nu_{b_{ij}}^q\right\}}{\max\left\{\nu_{a_{ij}}^q, \nu_{b_{ij}}^q\right\} + \min\left\{\nu_{a_{ij}}^q, \nu_{b_{ij}}^q\right\} - \max\left\{\nu_{a_{ij}}^q, \nu_{b_{ij}}^q\right\}\min\left\{\nu_{a_{ij}}^q, \nu_{b_{ij}}^q\right\}}\right)^{\frac{1}{q}}$$

$$= \left[\left\langle \left(\frac{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - 2\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{1 - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q \nu_{b_{ij}}^q}{\nu_{a_{ij}}^q + \nu_{b_{ij}}^q - \nu_{a_{ij}}^q \nu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right] A \boxplus_h B.$$

(ii) It can be proved similarly. □

3. On complement of q-rung orthopair fuzzy matrices

In this section, the complement of a q-rung orthopair fuzzy matrix is used to analyze the complementing nature of any system. Using the following results, we can study the complementing nature of a system with the help of the original q-rung orthopair fuzzy matrix. The operator complement obey De Morgan's law for the operations $\boxplus_h$ and $\boxdot_h$. This is established in the following theorem.

Theorem 9. *Let $A = [\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle]$, $B = [\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle]$ be any two q-ROFMs of the same size, then*

(i) $(A \boxplus_h B)^C = A^C \boxdot_h B^C$,
(ii) $(A \boxdot_h B)^C = A^C \boxplus_h B^C$,
(iii) $(A \boxplus_h B)^C \leq A^C \boxplus_h B^C$,
(iv) $(A \boxdot_h B)^C \geq A^C \boxdot_h B^C$.

Proof. (i) $A^C \boxdot_h B^C$

$$= \left[\left(\frac{\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{1 - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right]$$

$= (A \boxplus_h B)^C$.

(ii) $A^C \boxplus_h B^C$

$$= \left[\left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right]$$

$= (A \boxdot_h B)^C$.

(iii) $(A \boxplus_h B)^C$

$$= \left[\left(\frac{\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{1 - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right].$$

$A^C \boxplus_h B^C$

$$= \left[\left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right].$$

By Lemma 1, for all i, j

$$\left(\frac{\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \leq \left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}},$$

$$\left(\frac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{1 - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}} \geq \left(\frac{\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}}.$$

Hence, $(A \boxplus_h B)^C \leq A^C \boxplus_h B^C$.

(iv) It can be proved similarly. □

The following theorem is obvious.

Theorem 10. *Let* $A = [\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle]$, $B = [\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle]$ *be any two q-ROFMs of the same size, then*

(i) $(A^C \boxdot_h B^C)^C = A \boxplus_h B$,

(ii) $(A^C \boxplus_h B^C)^C = A \boxdot_h B$.

4. SCALAR MULTIPLICATION AND EXPONENTIATION OPERATIONS OF Q-RUNG ORTHOPAIR FUZZY MATRICES

We defined the following operations over Hamacher operations on q-ROFMs. In this section, we construct Hamacher scalar multiplication ($n._h A$) and Hamacher exponentiation ($A^{\wedge_h n}$) operations on q-rung orthopair fuzzy matrix A and investigate their algebraic properties.

Based on the Definition 1, Hamacher sum and Hamacher product over two q-ROFMs A and B are further indicated as the following operations.

Theorem 11. *If n is any positive integer and A is a PFM, then the Hamacher scalar multiplication operation* $(._h)$ *is*

$$n._h A = \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right], \tag{1}$$

where $n._h A = \underbrace{A \boxplus_h ... \boxplus_h A}_{n}$.

Proof. Mathematical induction can be used to prove that the above equation (1) holds for all positive integer n. The equation (1) is called $P(n)$. Using the above Definition 1, Hamacher sum $A \boxplus_h B$ we have:

$$\begin{aligned} A._h A &= \left[\left\langle \left(\frac{\mu^q_{a_{ij}}+\mu^q_{a_{ij}}-2\mu^q_{a_{ij}}\mu^q_{a_{ij}}}{1-\mu^q_{a_{ij}}\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}\nu^q_{a_{ij}}}{\nu^q_{a_{ij}}+\nu^q_{a_{ij}}-\nu^q_{a_{ij}}\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right] \\ &= \left[\left\langle \left(\frac{2\mu^q_{a_{ij}}-2a^q_{ij}}{1-a^q_{ij}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{2\nu^q_{a_{ij}}-\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right] \\ &= \left[\left\langle \left(\frac{2\mu^q_{a_{ij}}(1-\mu^q_{a_{ij}})}{1-a^q_{ij}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{\nu^q_{a_{ij}}(2-\nu^q_{a_{ij}})}\right)^{\frac{1}{q}} \right\rangle\right] \\ &= \left[\left\langle \left(\frac{2\mu^q_{a_{ij}}(1-\mu^q_{a_{ij}})}{(1-\mu^q_{a_{ij}})(1+\mu^q_{a_{ij}})}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{(2-\nu^q_{a_{ij}})}\right)^{\frac{1}{q}} \right\rangle\right] \\ &= \left[\left\langle \left(\frac{2\mu^q_{a_{ij}}}{1+\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{(2-\nu^q_{a_{ij}})}\right)^{\frac{1}{q}} \right\rangle\right]. \end{aligned}$$

$$2._h A = \left[\left\langle \left(\frac{2\mu_{a_{ij}}^q}{1+(2-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{2-(2-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

since $\mu_{a_{ij}}^q = (2-1)\mu_{a_{ij}}^q$,

$$n._h A = \left[\left\langle \left(\frac{n\mu_{a_{ij}}^q}{1+(n-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{n-(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$P(n)$ holds.

Suppose that equation (1) holds for $n = m$, i.e.,

$$\begin{aligned} m._h A &= \underbrace{A \boxplus_h \ldots \boxplus_h A}_{m} \\ &= \left[\left\langle \left(\frac{m\mu_{a_{ij}}^q}{1+(m-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{m-(m-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right]. \end{aligned}$$

Then,

$$\begin{aligned} &(m+1)._h A = ((m._h A) \boxplus_h A) \\ &= \left[\left(\frac{\frac{m\mu_{a_{ij}}^q}{1+(m-1)\mu_{a_{ij}}^q} + \mu_{a_{ij}}^q - 2\frac{m\mu_{a_{ij}}^q}{1+(m-1)\mu_{a_{ij}}^q}.\mu_{a_{ij}}^q}{1 - \frac{m\mu_{a_{ij}}^q}{1+(m-1)\mu_{a_{ij}}^q}.\mu_{a_{ij}}^q}\right)^{\frac{1}{q}},\right. \\ &\qquad \left.\left(\frac{\frac{\nu_{a_{ij}}^q}{m-(m-1)\nu_{a_{ij}}^q}.\nu_{a_{ij}}^q}{\frac{\nu_{a_{ij}}^q}{m-(m-1)\nu_{a_{ij}}^q} + \nu_{a_{ij}}^q - \frac{\nu_{a_{ij}}^q}{m-(m-1)\nu_{a_{ij}}^q}.\nu_{a_{ij}}^q}\right)^{\frac{1}{q}}\right] \\ &= \left[\left\langle \left(\frac{\mu_{a_{ij}}^q(m+1)(1-\mu_{a_{ij}}^q)}{(1+m\mu_{a_{ij}}^q)(1-\mu_{a_{ij}}^q)}\right)^{\frac{1}{q}}, \left(\frac{(\nu_{a_{ij}}^q)^2}{\nu_{a_{ij}}^q(m+1-m\nu_{a_{ij}}^q)}\right)^{\frac{1}{q}} \right\rangle\right] \\ &= \left[\left\langle \left(\frac{(m+1)\mu_{a_{ij}}^q}{1+m\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{m+1-m\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right] \\ &= \left[\left\langle \left(\frac{(m+1)\mu_{a_{ij}}^q}{1+[(m+1)-1]\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{m+1-[(m+1)-1]\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right]. \end{aligned}$$

So, when $n = m + 1$,

$$n._h A = \underbrace{A \boxplus_h \ldots \boxplus_h A}_{n}$$

$$= \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}} \right)^{\frac{1}{q}} \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

also holds.

Using the induction hypothesis that $P(n)$ holds for any positive integer n. □

Theorem 12. *If n is any positive integer and A is a PFM, then the Hamacher exponentiation operation $(\wedge_h)$ is*

$$A^{\wedge_h n} = \left[\left\langle \left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{n\nu^q_{a_{ij}}}{1+(n-1)\nu^q_{a_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right], \tag{2}$$

where $A^{\wedge_h n} = \overbrace{A \boxdot_h \dots \boxdot_h A}^{n}$.

Proof. Mathematical induction can be used to prove that the above equation (2) holds for all positive integer n. The equation (2) is called $P(n)$. Using the above Definition 1, Hamacher product $A \boxdot_h B$ we have

$$\begin{aligned}
A^{\wedge_h A} &= \left[\left\langle \left(\frac{\mu^q_{a_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{a_{ij}} - \mu^q_{a_{ij}}\mu^q_{a_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}} + \nu^q_{a_{ij}} - 2\nu^q_{a_{ij}}\nu^q_{a_{ij}}}{1-\nu^q_{a_{ij}}\nu^q_{a_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right] \\
&= \left[\left\langle \left(\frac{\mu^q_{a_{ij}}}{2\mu^q_{a_{ij}} - \mu^q_{a_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{2\nu^q_{a_{ij}} - 2\nu^q_{a_{ij}}}{1-\nu^q_{a_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right] \\
&= \left[\left\langle \left(\frac{\mu^q_{a_{ij}}}{\mu^q_{a_{ij}}(2-\mu^q_{a_{ij}})} \right)^{\frac{1}{q}}, \left(\frac{2\nu^q_{a_{ij}}(1-\nu^q_{a_{ij}})}{1-\nu^q_{a_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right] \\
&= \left[\left\langle \left(\frac{\mu^q_{a_{ij}}}{(2-\mu^q_{a_{ij}})} \right)^{\frac{1}{q}}, \left(\frac{2\nu^q_{a_{ij}}(1-\nu^q_{a_{ij}})}{(1-\nu^q_{a_{ij}})(1+\nu^q_{a_{ij}})} \right)^{\frac{1}{q}} \right\rangle \right] \\
&= \left[\left\langle \left(\frac{\mu^q_{a_{ij}}}{(2-\mu^q_{a_{ij}})} \right)^{\frac{1}{q}}, \left(\frac{2\nu^q_{a_{ij}}}{1+\nu^q_{a_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right].
\end{aligned}$$

$$A^{\wedge_h q} = \left[\left\langle \left(\frac{\mu^q_{a_{ij}}}{2-(2-1)\mu^q_{a_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{2\nu^q_{a_{ij}}}{1+(2-1)\nu^q_{a_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

since $\mu^q_{a_{ij}} = (2-1)\mu^q_{a_{ij}}$,

$$A^{\wedge_h n} = \left[\left\langle \left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{n\nu^q_{a_{ij}}}{1+(n-1)\nu^q_{a_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$P(n)$ holds.

Suppose that equation (2) holds for $n = m$, i.e.,

$$A^{\wedge_h m} = \overbrace{A \boxdot_h \ldots \boxdot_h A}^{m}$$
$$= \left[\left\langle \left(\frac{\mu_{a_{ij}}^q}{m-(m-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{m\nu_{a_{ij}}^q}{1+(m-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right].$$

So, when $n = m + 1$,

$$A^{\wedge_h m+1} = \left[\left\langle \left(\frac{{\mu_{a_{ij}}}^2}{m+1-[(m+1)-1]{\mu_{a_{ij}}}^2}\right)^{\frac{1}{q}}, \left(\frac{(m+1)\nu_{a_{ij}}^q}{1+[(m+1)-1]\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$A^{\wedge_h n} = \overbrace{A \boxdot_h \ldots \boxdot_h A}^{n}$$
$$= \left[\left\langle \left(\frac{\mu_{a_{ij}}^q}{n-(n-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

also holds.

Using the induction hypothesis that $P(n)$ holds for any positive integer n. □

Next, we prove the result of $(n._h A)$ and $(A^{\wedge_h n})$ are also q-ROFMs.

Theorem 13. *For any PFM A and for any positive integer n, then $(n._h A)$ and $(A^{\wedge_h n})$ are q-ROFMs.*

Proof. Since $0 \leq \mu_{a_{ij}}^q \leq 1$, $0 \leq \nu_{a_{ij}}^q \leq 1$, $0 \leq \mu_{a_{ij}}^q + \nu_{a_{ij}}^q \leq 1$ and $n > 1$, we have:

$$(n-1)\mu_{a_{ij}}^q > -1, \quad 1+(n-1)\mu_{a_{ij}}^q > 0,$$
$$n-(n-1)\nu_{a_{ij}}^q = (1-\nu_{a_{ij}}^q)n + \nu_{a_{ij}}^q > \nu_{a_{ij}}^q \geq 0.$$

Then, it is easy to get that

$$\left(\frac{n\mu_{a_{ij}}^q}{1+(n-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}} \geq 0, \quad \left(\frac{\nu_{a_{ij}}^q}{n-(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \geq 0.$$

Considering that

$$1+(n-1)\mu_{a_{ij}}^q = n\mu_{a_{ij}}^q + 1 - \mu_{a_{ij}}^q \geq n\mu_{a_{ij}}^q$$

and

$$n-(n-1)\nu_{a_{ij}}^q = \nu_{a_{ij}}^q + n(1-\nu_{a_{ij}}^q) \geq \nu_{a_{ij}}^q,$$

we get

$$\left(\frac{n\mu_{a_{ij}}^q}{1+(n-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}} \leq 1, \quad \left(\frac{\nu_{a_{ij}}^q}{n-(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \leq 1.$$

For $\mu^q_{a_{ij}} + \nu^q_{a_{ij}} \leq 1$, $0 \leq \nu^q_{a_{ij}} \leq 1 - \mu^q_{a_{ij}}$, it can be got that:

$$\begin{aligned}
&\left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} + \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \\
&= \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} + \left(\frac{1}{\frac{n}{\nu^q_{a_{ij}}}-(n-1)}\right)^{\frac{1}{q}} \\
&\leq \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} + \left(\frac{1}{\frac{n}{1-\mu^q_{a_{ij}}}-(n-1)}\right)^{\frac{1}{q}} \\
&= 1.
\end{aligned}$$

Thus,

$$0 \leq \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} \leq 1, \quad 0 \leq \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \leq 1,$$

so

$$\left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} + \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \leq 1.$$

Similarly, we can also get

$$0 \leq \left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} \leq 1, \quad 0 \leq \left(\frac{n\nu^q_{a_{ij}}}{1+(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \leq 1,$$

and

$$\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} + \left(\frac{n\nu^q_{a_{ij}}}{1+(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \leq 1.$$

Hence, $(n._h)A$ and $(A^{\wedge_h n})$ are q-ROFMs. □

Theorem 14. *Let $A = [\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle]$, $B = [\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle]$ be any two q-ROFMs of the same size and for any positive integers n, n_1, n_2.*

(i) $(n_1._h A) \boxplus_h (n_2._h A) = (n_1 + n_2)._h A$,
(ii) $(n._h A) \boxplus_h (n._h B) = n._h(A \boxplus_h B)$,
(iii) $A^{\wedge_h n_1} \boxdot_h A^{\wedge_h n_2} = A^{\wedge_h (n_1+n_2)}$,
(iv) $A^{\wedge_h n} \boxdot_h B^{\wedge_h n} = (A \boxdot_h B)^{\wedge_h n}$,
(v) $n_2._h(n_1._h A) = (n_1 n_2)._h A$,
(vi) $(A^{\wedge_h n_1})^{\wedge_h n_2} = A^{\wedge_h (n_1 n_2)}$.

Proof. In the following, we shall prove (i), (ii), (v) and (iii), (iv), (vi) can be proved analogously.

(i) By equations (1) and (2), we have:

$$\begin{aligned} n_1 \cdot_h A &= \left[\left\langle \left(\frac{n_1 \mu_{a_{ij}}^q}{1+(n_1-1)\mu_{a_{ij}}^q} \right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{n_1-(n_1-1)\nu_{a_{ij}}^q} \right)^{\frac{1}{q}} \right\rangle \right] \\ &= \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle \right], \\ n_2 \cdot_h A &= \left[\left\langle \left(\frac{n_2 \mu_{a_{ij}}^q}{1+(n_2-1)\mu_{a_{ij}}^q} \right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{n_2-(n_2-1)\nu_{a_{ij}}^q} \right)^{\frac{1}{q}} \right\rangle \right] \\ &= \left[\left\langle \mu_{c_{ij}}, \nu_{c_{ij}} \right\rangle \right], \end{aligned}$$

$$B \boxplus_h C = (n_1 \cdot_h A) \boxplus_h (n_2 \cdot_h A),$$

where

$$B \boxplus_h C = \left[\left\langle \left(\frac{\mu_{b_{ij}}^q + \mu_{c_{ij}}^q - 2\mu_{b_{ij}}^q \mu_{c_{ij}}^q}{1-\mu_{b_{ij}}^q \mu_{c_{ij}}^q} \right)^{\frac{1}{q}}, \left(\frac{\nu_{b_{ij}}^q \nu_{c_{ij}}^q}{\nu_{b_{ij}}^q + \nu_{c_{ij}}^q - \nu_{b_{ij}}^q \nu_{c_{ij}}^q} \right)^{\frac{1}{q}} \right\rangle \right].$$

We can further get:

$$\begin{aligned} &\left(\frac{\mu_{b_{ij}}^q + \mu_{c_{ij}}^q - 2\mu_{b_{ij}}^q \mu_{c_{ij}}^q}{1-\mu_{b_{ij}}^q \mu_{c_{ij}}^q} \right)^{\frac{1}{q}} \\ &= \left(\frac{\frac{n_1 \mu_{a_{ij}}^q}{1+(n_1-1)\mu_{a_{ij}}^q} + \frac{n_2 \mu_{a_{ij}}^q}{1+(n_2-1)\mu_{a_{ij}}^q} - 2\frac{n_1 \mu_{a_{ij}}^q}{1+(n_1-1)\mu_{a_{ij}}^q}\frac{n_2 \mu_{a_{ij}}^q}{1+(n_2-1)\mu_{a_{ij}}^q}}{1-\frac{n_1 \mu_{a_{ij}}^q}{1+(n_1-1)\mu_{a_{ij}}^q}\frac{n_2 \mu_{a_{ij}}^q}{1+(n_2-1)\mu_{a_{ij}}^q}} \right)^{\frac{1}{q}} \\ &= \left(\frac{n_1 \mu_{a_{ij}}^q (1+n_2 \mu_{a_{ij}}^q - \mu_{a_{ij}}^q) + n_2 \mu_{a_{ij}}^q (1+n_1 \mu_{a_{ij}}^q - \mu_{a_{ij}}^q) - 2n_1 \mu_{a_{ij}}^q n_2 \mu_{a_{ij}}^q}{(1+n_1 \mu_{a_{ij}}^q - \mu_{a_{ij}}^q)(1+n_2 \mu_{a_{ij}}^q - \mu_{a_{ij}}^q) - n_1 \mu_{a_{ij}}^q n_2 \mu_{a_{ij}}^q} \right)^{\frac{1}{q}} \\ &= \left(\frac{(n_1+n_2)\mu_{a_{ij}}^q - (n_1+n_2) a_{ij}^q}{1+(n_1+n_2-2)\mu_{a_{ij}}^q - (n_1+n_2-1) a_{ij}^q} \right)^{\frac{1}{q}} \\ &= \left(\frac{(n_1+n_2)\mu_{a_{ij}}^q (1-\mu_{a_{ij}}^q)}{(1+(n_1+n_2-1)\mu_{a_{ij}}^q)(1-\mu_{a_{ij}}^q)} \right)^{\frac{1}{q}}, \end{aligned}$$

$$n_1 \cdot_h A = \left(\frac{(n_1+n_2)\mu_{a_{ij}}^q}{1+(n_1+n_2-1)\mu_{a_{ij}}^q} \right)^{\frac{1}{q}}$$

and

$$\left(\frac{\nu_{b_{ij}}^q \nu_{c_{ij}}^q}{\nu_{b_{ij}}^q + \nu_{c_{ij}}^q - \nu_{b_{ij}}^q \nu_{c_{ij}}^q} \right)^{\frac{1}{q}}$$

$$= \left(\frac{\frac{\nu_{a_{ij}}^q}{n_1-(n_1-1)\nu_{a_{ij}}^q} \frac{\nu_{a_{ij}}^q}{n_2-(n_2-1)\nu_{a_{ij}}^q}}{\frac{\nu_{a_{ij}}^q}{n_1-(n_1-1)\nu_{a_{ij}}^q} + \frac{\nu_{a_{ij}}^q}{n_2-(n_2-1)\nu_{a_{ij}}^q} - \frac{\nu_{a_{ij}}^q}{n_1-(n_1-1)\nu_{a_{ij}}^q} \frac{\nu_{a_{ij}}^q}{n_2-(n_2-1)\nu_{a_{ij}}^q}} \right)^{\frac{1}{q}}$$

$$= \left(\frac{\nu_{a_{ij}}^q \nu_{a_{ij}}^q}{\nu_{a_{ij}}^q (n_2 + \nu_{a_{ij}}^q - n_2\nu_{a_{ij}}^q) + \nu_{a_{ij}}^q (n_1 + \nu_{a_{ij}}^q - n_1\nu_{a_{ij}}^q) - \nu_{a_{ij}}^q \nu_{a_{ij}}^q} \right)^{\frac{1}{q}}$$

$$= \left(\frac{\nu_{a_{ij}}^q}{(n_2 + \nu_{a_{ij}}^q - n_2\nu_{a_{ij}}^q) + (n_1 + \nu_{a_{ij}}^q - n_1\nu_{a_{ij}}^q) - \nu_{a_{ij}}^q} \right)^{\frac{1}{q}},$$

$$n_2 \cdot_h A = \left(\frac{\nu_{a_{ij}}^q}{(n_1+n_2) - (n_1+n_2-1)\nu_{a_{ij}}^q} \right)^{\frac{1}{q}}.$$

Since

$$(n_1+n_2) \cdot_h A = \left[\left\langle \left(\frac{(n_1+n_2)\mu_{a_{ij}}^q}{1+(n_1+n_2-1)\mu_{a_{ij}}^q} \right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{(n_1+n_2) - (n_1+n_2-1)\nu_{a_{ij}}^q} \right)^{\frac{1}{q}} \right\rangle \right],$$

we can finally get $(n_1 \cdot_h A) \boxplus_h (n_2 \cdot_h A) = (n_1+n_2) \cdot_h A$.

(ii) By equations (1) and (2), we have:

$$\begin{aligned}
n \cdot_h A &= \left[\left\langle \left(\frac{n\mu_{a_{ij}}^q}{1+(n-1)\mu_{a_{ij}}^q} \right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q}{n-(n-1)\nu_{a_{ij}}^q} \right)^{\frac{1}{q}} \right\rangle \right] \\
&= \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle \right], \\
n \cdot_h B &= \left[\left\langle \left(\frac{n\mu_{b_{ij}}^q}{1+(n-1)\mu_{b_{ij}}^q} \right)^{\frac{1}{q}}, \left(\frac{\nu_{b_{ij}}^q}{n-(n-1)\nu_{b_{ij}}^q} \right)^{\frac{1}{q}} \right\rangle \right] \\
&= \left[\left\langle \mu_{c_{ij}}, \nu_{c_{ij}} \right\rangle \right], \\
B \boxplus_h C &= (n \cdot_h A) \boxplus_h (n \cdot_h B), \\
B \boxplus_h C &= \left[\left\langle \left(\frac{\mu_{b_{ij}}^q + \mu_{c_{ij}}^q - 2\mu_{b_{ij}}^q \mu_{c_{ij}}^q}{1 - \mu_{b_{ij}}^q \mu_{c_{ij}}^q} \right)^{\frac{1}{q}}, \left(\frac{\nu_{b_{ij}}^q \nu_{c_{ij}}^q}{\nu_{b_{ij}}^q + \nu_{c_{ij}}^q - \nu_{b_{ij}}^q \nu_{c_{ij}}^q} \right)^{\frac{1}{q}} \right\rangle \right].
\end{aligned}$$

We can further get:

$$\left(\frac{\mu_{b_{ij}}^q + \mu_{c_{ij}}^q - 2\mu_{b_{ij}}^q \mu_{c_{ij}}^q}{1 - \mu_{b_{ij}}^q \mu_{c_{ij}}^q} \right)^{\frac{1}{q}}$$

$$= \left(\frac{\frac{n\mu_{a_{ij}}^q}{1+(n-1)\mu_{a_{ij}}^q} + \frac{n\mu_{b_{ij}}^q}{1+(n-1)\mu_{b_{ij}}^q} - 2\frac{n\mu_{a_{ij}}^q}{1+(n-1)\mu_{a_{ij}}^q}\frac{n\mu_{b_{ij}}^q}{1+(n-1)\mu_{b_{ij}}^q}}{1 - \frac{n\mu_{a_{ij}}^q}{1+(n-1)\mu_{a_{ij}}^q}\frac{n\mu_{b_{ij}}^q}{1+(n-1)\mu_{b_{ij}}^q}} \right)^{\frac{1}{q}}$$

$$= \left(\frac{n\mu^q_{a_{ij}}(1+n\mu^q_{b_{ij}}-\mu^q_{b_{ij}}) + n\mu^q_{b_{ij}}(1+n\mu^q_{a_{ij}}-\mu^q_{a_{ij}}) - 2n\mu^q_{a_{ij}}n\mu^q_{b_{ij}}}{(1+n\mu^q_{b_{ij}}-\mu^q_{b_{ij}})(1+n\mu^q_{a_{ij}}-\mu^q_{a_{ij}}) - n\mu^q_{a_{ij}}n\mu^q_{b_{ij}}} \right)^{\frac{1}{q}},$$

$$(n._h A) = \left(\frac{n\mu^q_{a_{ij}} + n\mu^q_{b_{ij}} - 2n\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{1+(n-1)(\mu^q_{a_{ij}}+\mu^q_{b_{ij}}) - (2n-1)\mu^q_{a_{ij}}\mu^q_{b_{ij}}} \right)^{\frac{1}{q}}$$

and

$$\left(\frac{\nu^q_{b_{ij}}\nu^q_{c_{ij}}}{\nu^q_{b_{ij}} + \nu^q_{c_{ij}} - \nu^q_{b_{ij}}\nu^q_{c_{ij}}} \right)^{\frac{1}{q}}$$

$$= \left(\frac{\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\frac{\nu^q_{b_{ij}}}{n-(n-1)\nu^q_{b_{ij}}}}{\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}} + \frac{\nu^q_{b_{ij}}}{n-(n-1)\nu^q_{b_{ij}}} - \frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\frac{\nu^q_{b_{ij}}}{n-(n-1)\nu^q_{b_{ij}}}} \right)^{\frac{1}{q}}$$

$$= \left(\frac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{\nu^q_{a_{ij}}(n+\nu^q_{b_{ij}}-n\nu^q_{b_{ij}}) + \nu^q_{b_{ij}}(n+\nu^q_{a_{ij}}-n\nu^q_{a_{ij}}) - \nu^q_{a_{ij}}\nu^q_{b_{ij}}} \right)^{\frac{1}{q}},$$

$$(n._h B) = \left(\frac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{n(\nu^q_{a_{ij}}+\nu^q_{b_{ij}}) - (2n-1)\nu^q_{a_{ij}}\nu^q_{b_{ij}}} \right)^{\frac{1}{q}}.$$

Thus,

$$n._h(A \boxplus_h B) = \left[\left(\frac{n\frac{\mu^q_{a_{ij}}+\mu^q_{b_{ij}}-2\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{1-\mu^q_{a_{ij}}\mu^q_{b_{ij}}}}{1+(n-1)\frac{\mu^q_{a_{ij}}+\mu^q_{b_{ij}}-2\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{1-\mu^q_{a_{ij}}\mu^q_{b_{ij}}}} \right)^{\frac{1}{q}}, \right.$$

$$(3) \qquad \left. \left(\frac{\frac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{\nu^q_{a_{ij}}+\nu^q_{b_{ij}}-\nu^q_{a_{ij}}\nu^q_{b_{ij}}}}{n-(n-1)\frac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{\nu^q_{a_{ij}}+\nu^q_{b_{ij}}-\nu^q_{a_{ij}}\nu^q_{b_{ij}}}} \right)^{\frac{1}{q}} \right]$$

$$= \left(\frac{n\mu^q_{a_{ij}} + n\mu^q_{b_{ij}} - 2n\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{1+(n-1)(\mu^q_{a_{ij}}+\mu^q_{b_{ij}}) - (2n-1)\mu^q_{a_{ij}}\mu^q_{b_{ij}}} \right)^{\frac{1}{q}},$$

$$\left(\frac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{n(\nu^q_{a_{ij}}+\nu^q_{b_{ij}}) - (2n-1)\nu^q_{a_{ij}}\nu^q_{b_{ij}}} \right)^{\frac{1}{q}}.$$

Comparing above results, we can finally get

$$(n._h A) \boxplus_h (n._h B) = n._h(A \boxplus_h B).$$

(v) By equations (1) and (2), we have:

$$n_1{\cdot_h}A = \left[\left\langle\left(\frac{n_1\mu^q_{a_{ij}}}{1+(n_1-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n_1-(n_1-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right] = \left[\left\langle\mu_{b_{ij}}, \nu_{b_{ij}}\right\rangle\right],$$

$$n_2{\cdot_h}(n_1{\cdot_h}A) = \left[\left\langle\left(\frac{n_2\mu_{b_{ij}}}{1+(n_2-1)\mu_{b_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu_{b_{ij}}}{n_2-(n_2-1)\nu_{b_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right].$$

We can further get:

$$\left(\frac{n_2\mu_{b_{ij}}}{1+(n_2-1)\mu_{b_{ij}}}\right)^{\frac{1}{q}} = \left(\frac{n_2\dfrac{n_1\mu^q_{a_{ij}}}{1+(n_1-1)\mu^q_{a_{ij}}}}{1+(n_2-1)\dfrac{n_1\mu^q_{a_{ij}}}{1+(n_1-1)\mu^q_{a_{ij}}}}\right)^{\frac{1}{q}} = \left(\frac{n_1n_2\mu^q_{a_{ij}}}{1+(n_1n_2-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}},$$

and

$$\left(\frac{\nu_{b_{ij}}}{n_2-(n_2-1)\nu_{b_{ij}}}\right)^{\frac{1}{q}} = \left(\frac{\dfrac{\nu^q_{a_{ij}}}{n_1-(n_1-1)\nu^q_{a_{ij}}}}{n_2-(n_2-1)\dfrac{\nu^q_{a_{ij}}}{n_1-(n_1-1)\nu^q_{a_{ij}}}}\right)^{\frac{1}{q}} = \left(\frac{\nu^q_{a_{ij}}}{n_1n_2-(n_1n_2-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}.$$

Since

$$n_1(n_2{\cdot_h})A = \left[\left\langle\left(\frac{n_1n_2\mu^q_{a_{ij}}}{1+(n_1n_2-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n_1n_2-(n_1n_2-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

we can finally get $n_2{\cdot_h}(n_1{\cdot_h}A) = (n_1n_2){\cdot_h}A$. □

Theorem 15. *Let $A = [\langle\mu_{a_{ij}}, \nu_{a_{ij}}\rangle]$, $B = [\langle\mu_{b_{ij}}, \nu_{b_{ij}}\rangle]$ be any two q-ROFMs of the same size and for any positive integer n.*

(i) $n.{_h}(A \wedge B) = (n.{_h}A) \wedge (n.{_h}B)$,
(ii) $n.{_h}(A \vee B) = (n.{_h}A) \vee (n.{_h}B)$,
(iii) $(A \wedge B)^{\wedge_h n} = A^{\wedge_h n} \wedge B^{\wedge_h n}$,
(iv) $(A \vee B)^{\wedge_h n} = A^{\wedge_h n} \vee B^{\wedge_h n}$.

Proof. In the following, we shall prove (ii), (iv) and (i), (iii) can be proved analogously.

(i) Since

$$(A \wedge B) = \left[\left\langle \min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right\rangle\right],$$

$$n._h(A \wedge B) = \left[\left\langle \mu_{c_{ij}}, \nu_{c_{ij}}\right\rangle\right], n._hA = \left[\left\langle \mu_{d_{ij}}, \nu_{d_{ij}}\right\rangle\right],$$

$$n._hB = \left[\left\langle \mu_{e_{ij}}, \nu_{e_{ij}}\right\rangle\right],$$

where

$$\mu_{c_{ij}} = \left(\frac{n(\min\left\{\mu^q_{a_{ij}}, \mu^q_{b_{ij}}\right\})}{1+(n-1)(\min\left\{\mu^q_{a_{ij}}, \mu^q_{b_{ij}}\right\})}\right)^{\frac{1}{q}},$$

$$\nu_{c_{ij}} = \left(\frac{(\max\left\{\nu^q_{a_{ij}}, \nu^q_{b_{ij}}\right\})}{n-(n-1)(\max\left\{\nu^q_{a_{ij}}, \nu^q_{b_{ij}}\right\})}\right)^{\frac{1}{q}},$$

we have

$$\begin{aligned}\mu_{c_{ij}} &= \left(\frac{n(\min\left\{\mu^q_{a_{ij}}, \mu^q_{b_{ij}}\right\})}{1+(n-1)(\min\left\{\mu^q_{a_{ij}}, \mu^q_{b_{ij}}\right\})}\right)^{\frac{1}{q}}\\ &= \min\left\{\left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{n\mu^q_{b_{ij}}}{1+(n-1)\mu^q_{b_{ij}}}\right)^{\frac{1}{q}}\right\},\\ &= \min\left\{\mu_{d_{ij}}, \mu_{e_{ij}}\right\},\end{aligned}$$

$$\begin{aligned}\nu_{c_{ij}} &= \left(\frac{(\max\left\{\nu^q_{a_{ij}}, \nu^q_{b_{ij}}\right\})}{n-(n-1)(\max\left\{\nu^q_{a_{ij}}, \nu^q_{b_{ij}}\right\})}\right)^{\frac{1}{q}}\\ &= \max\left\{\left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{b_{ij}}}{n-(n-1)\nu^q_{b_{ij}}}\right)^{\frac{1}{q}}\right\},\\ &= \max\left\{\nu_{\mu_{d_{ij}}}, \nu_{e_{ij}}\right\}.\end{aligned}$$

Comparing the above equations, we get:

$$\begin{aligned}(n._hA) \wedge (n._hB) &= \left[\left\langle \min\left\{\mu_{d_{ij}}, \nu_{d_{ij}}\right\}, \max\left\{\mu_{e_{ij}}, \nu_{e_{ij}}\right\}\right\rangle\right],\\ = &\left[\min\left\{\left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\},\right.\\ &\left.\max\left\{\left(\frac{n\mu^q_{b_{ij}}}{1+(n-1)\mu^q_{b_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{b_{ij}}}{n-(n-1)\nu^q_{b_{ij}}}\right)^{\frac{1}{q}}\right\}\right].\end{aligned}$$

Thus, we have $n._h(A \wedge B) = (n._hA) \wedge (n._hB)$.

(iii) Since

$$(A \wedge B) = \left[\left\langle \min\left\{\mu_{a_{ij}}, \mu_{b_{ij}}\right\}, \max\left\{\nu_{a_{ij}}, \nu_{b_{ij}}\right\}\right\rangle\right],$$

then

$$(A \wedge B)^{\wedge_h n} = \left[\left\langle \mu_{c_{ij}}, \nu_{c_{ij}}\right\rangle\right],$$

$$A^{\wedge_h n} = \left[\left\langle \mu_{d_{ij}}, \nu_{d_{ij}}\right\rangle\right], \quad B^{\wedge_h n} = \left[\left\langle \mu_{e_{ij}}, \nu_{e_{ij}}\right\rangle\right],$$

where

$$\mu_{c_{ij}} = \left(\frac{(\min\left\{\mu^q_{a_{ij}}, \mu^q_{b_{ij}}\right\})}{n-(n-1)(\min\left\{\mu^q_{a_{ij}}, \mu^q_{b_{ij}}\right\})}\right)^{\frac{1}{q}},$$

$$\nu_{c_{ij}} = \left(\frac{n(\max\left\{\nu^q_{a_{ij}}, \nu^q_{b_{ij}}\right\})}{1+(n-1)(\max\left\{\nu^q_{a_{ij}}, \nu^q_{b_{ij}}\right\})}\right)^{\frac{1}{q}},$$

we have

$$\begin{aligned}\mu_{c_{ij}} &= \left(\frac{(\min\left\{\mu^q_{a_{ij}}, \mu^q_{b_{ij}}\right\})}{n-(n-1)(\min\left\{\mu^q_{a_{ij}}, \mu^q_{b_{ij}}\right\})}\right)^{\frac{1}{q}} \\ &= \min\left\{\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\mu^q_{b_{ij}}}{n-(n-1)\mu^q_{b_{ij}}}\right)^{\frac{1}{q}}\right\}, \\ &= \min\left\{\mu_{d_{ij}}, \mu_{e_{ij}}\right\},\end{aligned}$$

$$\begin{aligned}\nu_{c_{ij}} &= \left(\frac{n(\max\left\{\nu^q_{a_{ij}}, \nu^q_{b_{ij}}\right\})}{1+(n-1)(\max\left\{\nu^q_{a_{ij}}, \nu^q_{b_{ij}}\right\})}\right)^{\frac{1}{q}} \\ &= \max\left\{\left(\frac{n\nu^q_{a_{ij}}}{1+(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{n\nu^q_{b_{ij}}}{1+(n-1)\nu^q_{b_{ij}}}\right)^{\frac{1}{q}}\right\}, \\ &= \max\left\{\nu_{d_{ij}}, \nu_{e_{ij}}\right\}.\end{aligned}$$

Comparing the above equations, we get:

$$\begin{aligned}A^{\wedge_h n} \wedge B^{\wedge_h n} &= \left[\left\langle \min\left\{\mu_{d_{ij}}, \nu_{d_{ij}}\right\}, \max\left\{\mu_{e_{ij}}, \nu_{e_{ij}}\right\}\right\rangle\right], \\ = \Bigg[&\min\left\{\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{n\nu^q_{a_{ij}}}{1+(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\}, \\ &\max\left\{\left(\frac{\mu^q_{b_{ij}}}{n-(n-1)\mu^q_{b_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{n\nu^q_{b_{ij}}}{1+(n-1)\nu^q_{b_{ij}}}\right)^{\frac{1}{q}}\right\}\Bigg].\end{aligned}$$

Hence,

$$(A \wedge B)^{\wedge_h n} = A^{\wedge_h n} \wedge B^{\wedge_h n}. \qquad \square$$

5. Necessity and possibility operators on q-rung orthopair fuzzy matrices

In this section, some properties of necessity and possibility operators on q-rung orthopair fuzzy matrices are verify.

Definition 4. For any q-rung orthopair fuzzy matrix A, the necessity ($\square$) and the possibility ($\lozenge$) operators are defined as follows:

$$\square A = \left[\left\langle \mu_{a_{ij}}, \left(1-\mu^q_{a_{ij}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$\lozenge A = \left[\left\langle \left(1-\nu^q_{a_{ij}}\right)^{\frac{1}{q}}, \nu_{a_{ij}} \right\rangle\right].$$

Theorem 16. *Let* $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right]$, $B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$ *be any two q-ROFMs of the same size, then*

(i) $\square(A \boxplus_h B) = \square A \boxplus_h \square B$,

(ii) $\lozenge(A \boxplus_h B) = \lozenge A \boxplus_h \lozenge B$.

Proof. (i) $\square(A \boxplus_h B)$

$$= \left[\left\langle \left(\frac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{1-\mu^q_{a_{ij}}\mu^q_{b_{ij}}}\right)^{\frac{1}{q}}, \left(1 - \frac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{1-\mu^q_{a_{ij}}\mu^q_{b_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}}\mu^q_{b_{ij}}}{1-\mu^q_{a_{ij}}\mu^q_{b_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{(1-\mu^q_{a_{ij}})(1-\mu^q_{b_{ij}})}{1-\mu^q_{a_{ij}}\mu^q_{b_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$= \square A \boxplus_h \square B.$

(ii) $\lozenge(A \boxplus_h B)$

$$= \left[\left\langle \left(1 - \frac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}}\nu^q_{b_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}}\nu^q_{b_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}}\nu^q_{b_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}\nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}}\nu^q_{b_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$= \lozenge A \boxplus_h \lozenge B.$ $\square$

Theorem 17. *Let* $A = \left[\left\langle \mu_{a_{ij}}, \nu_{a_{ij}} \right\rangle\right]$, $B = \left[\left\langle \mu_{b_{ij}}, \nu_{b_{ij}} \right\rangle\right]$ *be any two q-ROFMs of the same size, then*

(i) $\square(A \boxdot_h B) = \square A \boxdot_h \square B$,

(ii) $\lozenge(A \boxdot_h B) = \lozenge A \boxdot_h \lozenge B$.

Proof. (i) $\Box(A \boxdot_h B)$

$$= \left[\left\langle \left(\frac{\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(1 - \frac{\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$$= \left[\left\langle \left(\frac{\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - 2\mu^q_{a_{ij}} \mu^q_{b_{ij}}}{\mu^q_{a_{ij}} + \mu^q_{b_{ij}} - \mu^q_{a_{ij}} \mu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$$= \Box A \boxdot_h \Box B.$$

(ii) $\Diamond(A \boxdot_h B)$

$$= \left[\left\langle \left(1 - \frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$$= \left[\left\langle \left(\frac{(1 - \nu^q_{a_{ij}})(1 - \nu^q_{b_{ij}})}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$$= \Diamond A \boxdot_h \Diamond B. \qquad \Box$$

Theorem 18. *Let* $A = \left[\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle\right]$, $B = \left[\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle\right]$ *be any two q-ROFMs of the same size, then*

(i) $\left(\Box(A^C \boxplus_h B^C)\right)^C = \Diamond A \boxdot_h \Diamond B,$

(ii) $\left(\Box(A^C \boxdot_h B^C)\right)^C = \Diamond A \boxplus_h \Diamond B.$

Proof. (i) $\Box(A^C \boxplus_h B^C)$

$$= \left[\left\langle \left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(1 - \frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$$= \left[\left\langle \left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{(1 - \nu^q_{a_{ij}})(1 - \nu^q_{b_{ij}})}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$$\left(\Box(A^C \boxplus_h B^C)\right)^C$$

$$= \left[\left\langle \left(\frac{(1 - \nu^q_{a_{ij}})(1 - \nu^q_{b_{ij}})}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - 2\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{1 - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$$= \Diamond A \boxdot_h \Diamond B.$$

(ii) $\Box(A^C \boxdot_h B^C)$

$$= \left[\left\langle \left(\frac{\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}}, \left(1 - \frac{\nu^q_{a_{ij}} \nu^q_{b_{ij}}}{\nu^q_{a_{ij}} + \nu^q_{b_{ij}} - \nu^q_{a_{ij}} \nu^q_{b_{ij}}} \right)^{\frac{1}{q}} \right\rangle \right],$$

$$= \left[\left\langle \left(\frac{\nu_{a_{ij}}^q \nu_{b_{ij}}^q}{\nu_{a_{ij}}^q + \nu_{b_{ij}}^q - \nu_{a_{ij}}^q \nu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q + \nu_{b_{ij}}^q - 2\nu_{a_{ij}}^q \nu_{b_{ij}}^q}{\nu_{a_{ij}}^q + \nu_{b_{ij}}^q - \nu_{a_{ij}}^q \nu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$\left(\Box(A^C \boxdot_h B^C)\right)^C$$

$$= \left[\left\langle \left(\frac{\nu_{a_{ij}}^q + \nu_{b_{ij}}^q - 2\nu_{a_{ij}}^q \nu_{b_{ij}}^q}{\nu_{a_{ij}}^q + \nu_{b_{ij}}^q - \nu_{a_{ij}}^q \nu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\nu_{a_{ij}}^q \nu_{b_{ij}}^q}{\nu_{a_{ij}}^q + \nu_{b_{ij}}^q - \nu_{a_{ij}}^q \nu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \Diamond A \boxplus_h \Diamond B. \qquad \Box$$

Theorem 19. *Let* $A = [\langle \mu_{a_{ij}}, \nu_{a_{ij}} \rangle]$, $B = [\langle \mu_{b_{ij}}, \nu_{b_{ij}} \rangle]$ *be any two q-ROFMs of the same size, then*

(i) $\left(\Diamond(A^C \boxplus_h B^C)\right)^C = \Box A \boxdot_h \Box B$,

(ii) $\left(\Diamond(A^C \boxdot_h B^C)\right)^C = \Box A \boxplus_h \Box B$.

Proof. (i) $\Diamond(A^C \boxplus_h B^C)$

$$= \left[\left\langle \left(1 - \frac{\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - 2\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$(\Diamond(A^C \boxplus_h B^C))^C$$

$$= \left[\left\langle \left(\frac{\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - 2\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \Box A \boxdot_h \Box B.$$

(ii) $\Diamond(A^C \boxdot_h B^C)$

$$= \left[\left\langle \left(1 - \frac{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - 2\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{1 - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - 2\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{1 - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{(1 - \mu_{a_{ij}}^q)(1 - \mu_{b_{ij}}^q)}{1 - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - 2\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{1 - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$(\Diamond(A^C \boxdot_h B^C))^C$$

$$= \left[\left\langle \left(\frac{\mu_{a_{ij}}^q + \mu_{b_{ij}}^q - 2\mu_{a_{ij}}^q \mu_{b_{ij}}^q}{1 - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{(1 - \mu_{a_{ij}}^q)(1 - \mu_{b_{ij}}^q)}{1 - \mu_{a_{ij}}^q \mu_{b_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \Box A \boxplus_h \Box B. \qquad \Box$$

Theorem 20. *For any q-ROFM A and for any positive integer n.*

(i) $\Box(n._h A) = n._h(\Box A)$,
(ii) $\Diamond(n._h A) = n._h(\Diamond A)$,
(iii) $\Box A^{\wedge_h n} = (\Box A)^{\wedge_h n}$,
(iv) $\Diamond A^{\wedge_h n} = (\Diamond A)^{\wedge_h n}$.

Proof. (i) $\Box(n._h A)$

$$= \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(1-\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(1-\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{1+(n-1)\mu^q_{a_{ij}}-n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{1+n\mu^q_{a_{ij}}-\mu^q_{a_{ij}}-n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$\Box(n._h A) = \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{1-\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$n._h(\Box A)$

$$= \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{1-\mu^q_{a_{ij}}}{n-(n-1)\left(1-\mu^q_{a_{ij}}\right)}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{1-\mu^q_{a_{ij}}}{n-(n-1)\left(1-\mu^q_{a_{ij}}\right)}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$n._h(\Box A) = \left[\left\langle \left(\frac{n\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{1-\mu^q_{a_{ij}}}{1+(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right].$$

Hence, $\Box(n._h A) = n._h(\Box A)$.

(ii) $\Diamond(n._h A)$

$$= \left[\left\langle \left(1-\left(\left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(1-\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle\left(\frac{n-(n-1)\nu^q_{a_{ij}}-\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$$= \left[\left\langle\left(\frac{n-n\nu^q_{a_{ij}}+\nu^q_{a_{ij}}-\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$$\Diamond(n._h A) = \left[\left\langle\left(\frac{n(1-\nu^q_{a_{ij}})}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$n._h(\Diamond A)$

$$= \left[\left\langle\left(\frac{n(\left(1-\nu^q_{a_{ij}}\right)^{\frac{1}{q}})^q}{1+(n-1)(\left(1-\nu^q_{a_{ij}}\right)^{\frac{1}{q}})^q}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$$= \left[\left\langle\left(\frac{n(1-\nu^q_{a_{ij}})}{1+(n-1)(1-\nu^q_{a_{ij}})}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$$n._h(\Diamond A) = \left[\left\langle\left(\frac{n(1-\nu^q_{a_{ij}})}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{\nu^q_{a_{ij}}}{n-(n-1)\nu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right].$$

Hence, $\Diamond(n._h A) = n._h(\Diamond A)$.

(iii) $\Box A^{\wedge_h n}$

$$= \left[\left\langle\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(1-\left(\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}}\right\rangle\right],$$

$$= \left[\left\langle\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(1-\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$$= \left[\left\langle\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{n-(n-1)\mu^q_{a_{ij}}-\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$$= \left[\left\langle\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{n-n\mu^q_{a_{ij}}+\mu^q_{a_{ij}}-\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$$= \left[\left\langle\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{n(1-\mu^q_{a_{ij}})}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}\right\rangle\right],$$

$(\Box A)^{\wedge_h n}$

$$= \left[\left\langle\left(\frac{\mu^q_{a_{ij}}}{n-(n-1)\mu^q_{a_{ij}}}\right)^{\frac{1}{q}}, \left(\frac{n(\left(1-\mu^q_{a_{ij}}\right)^{\frac{1}{q}})^q}{n-(n-1)(\left(1-\mu^q_{a_{ij}}\right)^{\frac{1}{q}})^q}\right)^{\frac{1}{q}}\right\rangle\right],$$

$$= \left[\left\langle \left(\frac{\mu_{a_{ij}}^q}{n-(n-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{n(1-\mu_{a_{ij}}^q)}{n-(n-1)(1-\mu_{a_{ij}}^q)}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$(\Box A)^{\wedge_h n} = \left[\left\langle \left(\frac{\mu_{a_{ij}}^q}{n-(n-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{n(1-\mu_{a_{ij}}^q)}{n-(n-1)\mu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right].$$

Hence, $\Box A^{\wedge_h n} = (\Box A)^{\wedge_h n}$.

(iv) $\Diamond A^{\wedge_h n}$

$$= \left[\left\langle \left(1-\left(\left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}}\right)^q\right)^{\frac{1}{q}}, \left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(1-\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{1+(n-1)\nu_{a_{ij}}^q-n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$\Diamond A^{\wedge_h n} = \left[\left\langle \left(\frac{1-\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$(\Diamond A)^{\wedge_h n}$

$$= \left[\left\langle \left(\frac{\left((1-\nu_{a_{ij}}^q)^{\frac{1}{q}}\right)^q}{1+(n-1)\left((1-\nu_{a_{ij}}^q)^{\frac{1}{q}}\right)^q}\right)^{\frac{1}{q}}, \left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$= \left[\left\langle \left(\frac{(1-\nu_{a_{ij}}^q)}{1+(n-1)\left(1-\nu_{a_{ij}}^q\right)}\right)^{\frac{1}{q}}, \left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right],$$

$$(\Diamond A)^{\wedge_h n} = \left[\left\langle \left(\frac{1-\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}}, \left(\frac{n\nu_{a_{ij}}^q}{1+(n-1)\nu_{a_{ij}}^q}\right)^{\frac{1}{q}} \right\rangle\right].$$

Hence, $\Diamond A^{\wedge_h n} = (\Diamond A)^{\wedge_h n}$, which completes the proof of this theorem. □

6. CONCLUSION

The work has extended the Hamacher operation results under q-rung orthopair fuzzy environment. In this paper, we have developed the Hamacher operations of q-rung orthopair fuzzy matrices and investigated their algebraic properties. We also proved that the set of all q-ROFMs with respect to Hamacher sum and Hamacher product form a commutative monoid.

A study of the algebraic structure of q-ROFMs with respect to Hamacher operations gives us a deep insight into the applications. Then, De Morgan's laws are verified. Furthermore, we constructed Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations on q-rung orthopair fuzzy matrix A and investigated their algebraic properties. Further, some properties of necessity and possibility operators on q-rung orthopair fuzzy matrices are verified. It is worth to point out that the proposed Hamacher operations over q-ROFMs will be applied to aggregating q-rung orthopair fuzzy information in the future.

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